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Finite amplitude axisymmetric convection between rigid rotating planes

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Abstract: This paper considers the nature of stationary axisymmetric convection in a rotating layer of fluid heated from below. The nonlinear amplitude equation for the case of rigid boundaries is derived, and it is found that for certain ranges of the speed of rotation and Prandtl number subcritical instability is possible. The outer solution is matched with the inner solution (which can be expressed in terms of Bessel functions) and is similar to that described by Brown and Stewartson (1978).

1. Introduction

The classical theory of Benard convection in an infinite fluid layer was considered by Chandrasekhar [2]; nonlinear effects were first investigated by Veronis [10] who assumed disturbances of uniform amplitude in a horizontally infinite region and used rectangular cartesian coordinates with stress-free upper and lower surfaces. Kuppers and Lortz [9] analysed the solution of the nonlinear steady state equations in an infinite horizontal rotating fluid layer. Daniels (1978) also used stress-free horizontal surfaces and derived the amplitude equation for two-dimensional rolls in a rotating system bounded by distant side walls, considering both stationary and overstable motion. He incorporated both spatial modulation and nonlinear effects. More recently, Daniels [6] studied the effect of centrifugal acceleration on axisymmetric convection in a shallow rotating cylinder or annulus. Here we consider the nonlinear structure of flow at the onset of convection in a rotating layer with rigid boundaries, when the exchange of stabilities occurs.

Our amplitude equation is derived in Section 3 and it is found that, for certain ranges of speed of rotation and Prandtl number of fluid, subcritical instability is possible, as in the model problem with stress-free boundaries considered by Veronis [10]. In Section 4 an asymptotic analysis for large Taylor numbers, $T \gg 1$, is carried out and used to provide a check on the numerical computation of the amplitude equation coefficient. The solution of Section 3 provides an, other solution, and this must be matched to a solution (which can be expressed in terms of Bessel functions) near the axis of rotation, in order to complete the analysis and determine the boundary condition at the origin for the amplitude function involved in the outer solution. It is found (Section 5) that this condition is essentially unaltered from that derived by Brown and Stewartson [1] for a nonrotating layer with stress-free boundaries, although their theory has to be

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modified to take into account both the rigid boundaries and the rotation of the layer. Thus, the main features of the Brown and Stewartson study apply to the rotating case also. Finally, subcritical and supercritical solutions of the amplitude equation are given in Section 6.

2. The nonlinear system

After applying the Bousinesq approximation the equations in cylindrical coordinates (r, ϕ, z), which govern the axisymmetric motion of a fluid with thermal diffusivity κ, kinematic viscosity ν and coefficient of the thermal expansion α, under gravity g, may be reduced to the dimensionless form

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0. \tag{2.1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z} - \frac{v^2}{r} = -\frac{\partial p}{\partial r} + \sigma T^{1/2} v + \sigma\left(\nabla^2 u - \frac{u}{r^2}\right), \tag{2.2}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + w\frac{\partial v}{\partial z} + \frac{uv}{r} = -\sigma T^{1/2} u + \sigma\left(\nabla^2 v + \frac{v}{r^2}\right), \tag{2.3}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z} = -\frac{\partial p}{\partial z} + \sigma(\nabla^2 w + \theta), \tag{2.4}$$

$$\frac{\partial \theta}{\partial t} + u\frac{\partial \theta}{\partial r} + w\frac{\partial \theta}{\partial z} = Rw + \nabla^2\theta, \tag{2.5}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}.$$

Length scales are made dimensionless with respect to the distance d between horizontal planes $z = 0$, $z = 1$. These are held at constant temperatures θ_0^* and θ_1^* with $\theta_1^* < \theta_0^*$. The velocity components u, v, $w(r, z, t)$, reduced pressure $p(r, z, t)$, the perturbation $\theta(r, z, t)$, from the basic conductive temperature field $\theta_0^* - z(\theta_0^* - \theta_1^*)$ and the time t are made dimensionless with respect to the scales κ/d, $\rho\kappa^2/d^2$, $\kappa\nu/\alpha g d^3$ and d^2/κ respectively, where ρ is the density of the fluid. In equations (2.1)–(2.5) the Taylor number T, Prandtl number σ, and Rayleigh number R are defined by

$$T = 4\Omega^2 d^4/\nu^2, \qquad \sigma = \nu/\kappa, \qquad R = \alpha g d^3(\theta_0^* - \theta_1^*)/\kappa\nu. \tag{2.6}$$

where Ω is the angular speed of rotation.

We shall assume the horizontal planes, $z = 0$, $z = 1$, to be rigid so that the boundary conditions are

$$u = v = w = \theta = 0 \quad \text{at } z = 0,\ z = 1. \tag{2.7}$$

3. Outer solution and amplitude equation

For consideration of slightly supercritical flows we set

$$R = R_c + \epsilon^2\beta, \tag{3.1}$$

where β is a constant factor introduced for convenience, R_c is the critical Rayleigh number and we shall assume $\epsilon \ll 1$. Away from the axis of rotation we use a multiple-scale method in which slow spatial and time variables are defined by

$$s = \epsilon r, \qquad \bar{\tau} = \epsilon^2 t. \tag{3.2}$$

We eliminate the pressure between (2.2)–(2.3), and away from the centre the solution may be expanded in the form

$$\begin{pmatrix} u \\ v \\ w \\ \theta \end{pmatrix}(r, z, t) = \epsilon \begin{pmatrix} u_1 \\ v_1 \\ w_1 \\ \theta_1 \end{pmatrix} + \epsilon^2 \begin{pmatrix} u_2 \\ v_2 \\ w_2 \\ \theta_2 \end{pmatrix} + \epsilon^3 \begin{pmatrix} u_3 \\ v_3 \\ w_3 \\ \theta_3 \end{pmatrix} + \dots . \tag{3.3}$$

On substituting (3.1)–(3.3) into the continuity equation (2.1), the vorticity equation and heat transfer equation, and equating terms of order ϵ, ϵ^2, ϵ^3 we obtain a set of linear partial differential equations,

$$\left.\begin{aligned} &u_{jr} + w_{jz} = \phi_{1j}, \\ &v_{jr} + v_{jzz} - T^{1/2}u_j = \phi_{2j}, \\ &\theta_{jrr} + \theta_{jzz} + R_c W_j = \phi_{3j}, \\ &u_{jrrz} + u_{jzzz} - w_{jrrr} - w_{jrzz} - \theta_{jr} + T^{1/2}v_{jz} = \phi_{4j}, \end{aligned}\right\} \tag{3.4}$$

$$u_j = v_j = \theta_j = w_j = 0 \quad \text{at } z = 0, 1; \quad j = 1, 2, 3, \tag{3.5}$$

where $u_{jr} = \partial u_j/\partial r$, $W_{jz} = \partial w_j/\partial z$, etc. and ϕ_{kj} ($k = 1, 2, 3, 4$ and $j = 1, 2, 3$) are functions to be defined below.

At order ϵ ($j = 1$) the functions ϕ_{kj} ($k = 1, 2, 3, 4$) are zero and the solutions for u_1, w_1, v_1, θ_1, which satisfy the boundary conditions (3.5), are given by

$$\begin{aligned} (u_1, v_1) &= \{A_0\, e^{i\alpha_c r} + A_0^*\, e^{-i\alpha_c r}\}(Df, k), \\ (w_1, \theta_1) &= i\{A_0\, e^{i\alpha_c r} - A_0^*\, e^{-i\alpha_c r}\}(-\alpha_c f, h), \end{aligned} \tag{3.6}$$

where f, k, h are real functions of z, A_0 $(s, \bar{\tau})$ is a complex amplitude function with conjugate A_0^*, $D = d/dz$ and α_c is the critical wave number to be determined below. Substitution of (3.6) into (3.4) leads to the basic linear system

$$\left.\begin{aligned} &D^2h - \alpha^2 h - \alpha R f = 0, \\ &D^2k - \alpha^2 k - T^{1/2} f = 0, \\ &D^4 f - 2\alpha^2 D^2 f + \alpha^4 f + \alpha h + T^{1/2} Dk = 0, \end{aligned}\right\} \tag{3.7}$$

to be solved subject to the conditions

$$h = f = Df = k = 0 \quad \text{at } z = 0, 1 \tag{3.8}$$

where $\alpha = \alpha_c$. The critical wave number α_c and Rayleigh number R_c are determined from the solution of (3.7)–(3.8) at which $dR/d\alpha = 0$. It can be shown that this condition is equivalent to

$$I = \int_0^1 \left(\alpha_c h^2/R_c - \alpha_c^2 k^2 - 2\alpha_c f D^2 f + 2\alpha_c^3 f^2 + hf\right) dz = 0. \tag{3.9}$$

From (3.7) and (3.8) we find that

$$L[h] = 0, \tag{3.10}$$

$$\left.\begin{aligned} &h = \mathrm{D}^2 h = 0,\\ &\mathrm{D}^3(\mathrm{D}^4 - 3\alpha^2\mathrm{D}^2 + R + 2\alpha^4)h = 0,\\ &\mathrm{D}(\mathrm{D}^2 - \alpha^2)h = 0, \end{aligned}\right\} \quad \text{at } z = 0, 1 \tag{3.11}$$

where $L = (\mathrm{D}^2 - \alpha^2)\{(\mathrm{D}^2 - \alpha^2)^3 - T\mathrm{D}^2 + \alpha^2 R\}$. Equation (3.10) and the boundary conditions (3.11) constitute an eigenvalue problem for R, for given α and T. The problem of determining the critical Rayleigh number for the onset of instability as stationary convection, at a given taylor number T, reduces to that of finding the lowest value of R as a function of α. The desired solution of (3.10)–(3.11) can be made unique by adding the normalization condition

$$h = 1 \quad \text{at } z = \tfrac{1}{2}. \tag{3.12}$$

In order to solve the system (3.10)–(3.11) a 4th-order Runge–Kutta scheme was used with starting values for h and its derivatives are specified at $z = 0$ and h and its derivatives calculated at equally spaced values of z up to $z = 1$. Two different step-sizes ($\delta z = 0.025, 0.0125$) were used to check the accuracy of the scheme.

Four linearly independent solutions $h^{(j)}$ ($j = 1, 2, 3, 4$) of (3.10), each satisfying the boundary conditions (3.11) at $z = 0$, are computed; then the function

$$h = h^{(1)} + a_1 h^{(2)} + a_2 h^{(3)} + a_3 h^{(4)}, \tag{3.13}$$

satisfies the conditions (3.11) at $z = 0$, for all values of a_j ($j = 1, 2, 3$). We now choose a_j such that the first three conditions of (3.11) are satisfied at $z = 1$. Then set

$$Q = D(D^2 - \alpha^2)h \quad \text{at } z = 1, \tag{3.14}$$

and the final boundary condition, $Q = 0$ at $z = 1$, leads to a relation between α and R for given T. To obtain $Q = 0$, Newton's method is applied in the form

$$R_{\text{new}} = R_{\text{old}} - Q(\partial Q/\partial R)^{-1}. \tag{3.15}$$

This iteration method thus provides the values of R on the neutral stability curve at given values of α and T. The lowest value of R and the associated value of α, at fixed T, provide the critical Rayleigh number and wave number for a given rotation rate. The profiles of h, f, k are shown in Fig. 1.

At order ϵ^2 ($j = 2$) the functions ϕ_{k2} ($k = 1, 2, \ldots, 4$) are found to be

$$\left.\begin{aligned} &\phi_{12} = -\mathrm{D}f\{e^{i\alpha_c r}B_1 + cc\},\\ &\phi_{22} = -i\alpha_c\{A_0^2 E_1\, e^{2i\alpha_c r} - (B_1 + \partial A_0/\partial s)k\, e^{i\alpha_c r}\} + cc,\\ &\phi_{32} = \alpha_c\{E_3 A_0^2 - 2E_4|A_0|^2 + (B_1 + \partial A_0/\partial s)\, e^{i\alpha_c r}h\} + cc,\\ &\phi_{42} = i\, e^{i\alpha_c r}\{(B_1 + 2\,\partial A_0/\partial s)(\alpha_c^3 f - \alpha_c \mathrm{D}^2 f) + \partial A_0/\partial s h + i\alpha_c E_2 A_0^2\, e^{2i\alpha_c r}\} + cc \end{aligned}\right\} \tag{3.16}$$

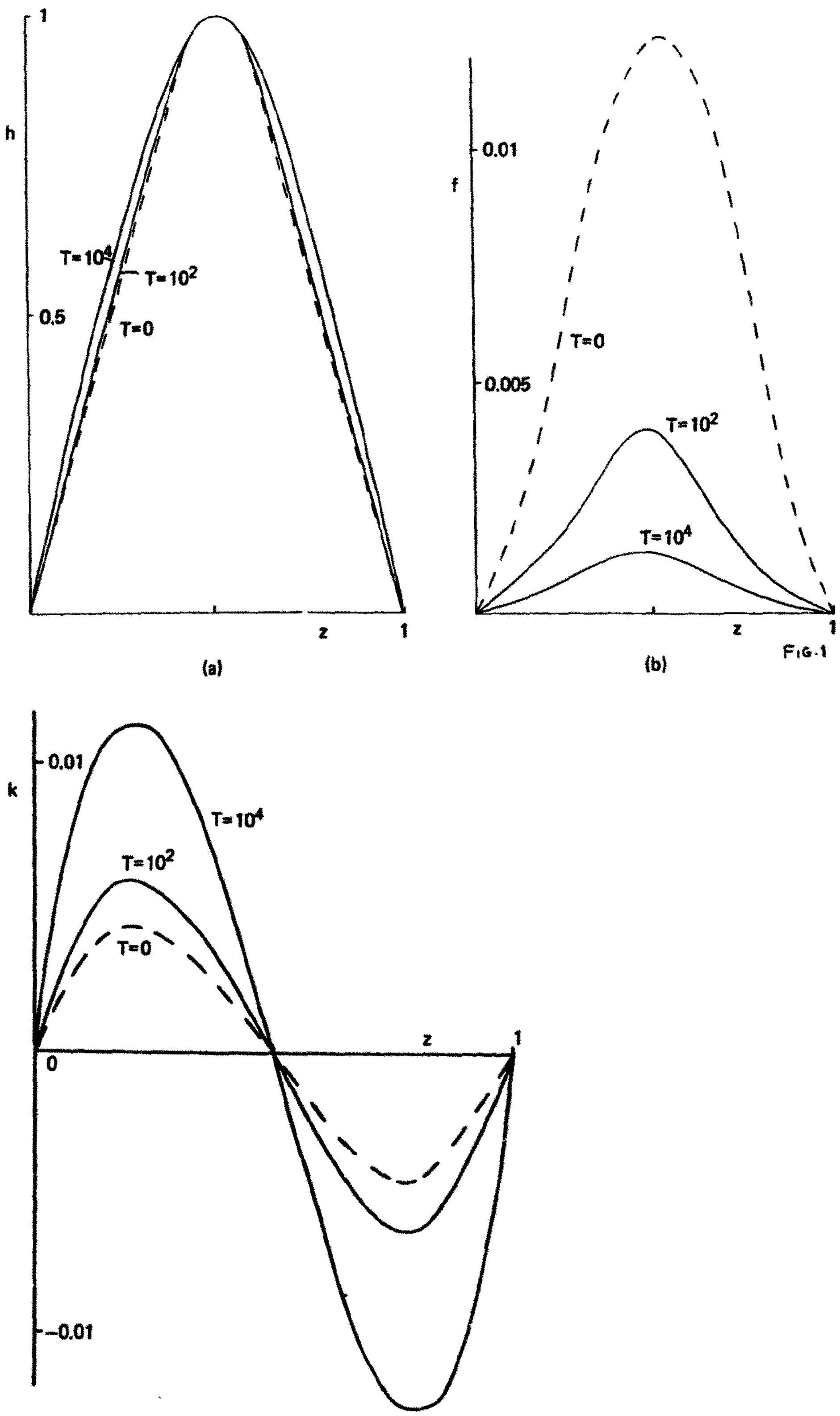

Fig. 1. (a) Temperature profiles for various rotation rates. (b) Vertical velocity component (w). (c) Azimuthal velocity component (v).

where cc denotes the complex conjugate, E_j $(j=1,\dots,4)$ are functions of z which are given in terms of the basic eigenfunctions and their derivatives with respect to the variable z, and $B_1=\partial A_0/\partial s+A_0/s$. The solutions for w_2, v_2, θ_2 contain components

$$(w_2, v_2, \theta_2)=(a_c F_1, \mathrm{i}K_1, -H_1)(s, z, \bar{\tau})\,\mathrm{e}^{\mathrm{i}\alpha_c}+\cdots. \tag{3.17}$$

and the functions F_1, K_1, H_1 satisfy equations of the form (3.7) but with the three zero right-hand sides replaced by

$$\left.\begin{aligned}\phi_1&=-\alpha_c\left(B_1+\frac{\partial A_0}{\partial s}\right)k+T^{1/2}\mathrm{D}fB_1/\alpha_c, \qquad \phi_2=-\alpha_c\left(B_1+\frac{\partial A_0}{\partial s}\right)h,\\ \phi_3&=\left(h+\alpha_c^2 f\right)\frac{\partial A_0}{\partial s}-(2\alpha_c\mathrm{D}f)\frac{\partial A_0}{\partial s}-(B_1/\alpha_c)\mathrm{D}^4 f+\alpha_c^3\left(B_1+\frac{\partial A_0}{\partial s}\right)f.\end{aligned}\right\} \tag{3.18}$$

Multiplication of the first of the equations (3.7) by H_1 and the corresponding equation for H_1 by h, subtraction and integration from $z=0$ to $z=1$, yields

$$-\alpha_c R_c\int_0^1 (F_1 h-fH_1)\,\mathrm{d}z=\int_0^1\phi_1 h\,\mathrm{d}z, \tag{3.19}$$

and from a similar treatment of the other equations,

$$-T^{1/2}\int_0^1\left(K\frac{\partial F_1}{\partial z}-K\mathrm{D}f\right)\mathrm{d}z=\int_0^1\phi_2 k\,\mathrm{d}z, \tag{3.20}$$

$$T^{1/2}\int_0^1\left(\frac{\partial K_1}{\partial z}f-F_1\mathrm{D}k\right)\mathrm{d}z+\alpha_c\int_0^1(H_1 f-hF_1)\,\mathrm{d}z=\int_0^1\phi_3 f\,\mathrm{d}z. \tag{3.21}$$

Combining these three results, we find that a consistent solution leads to the requirement that

$$\int_0^1\left(R_c^{-1}\phi_1 h-k\phi_2-f\phi_3\right)\mathrm{d}z=0. \tag{3.22}$$

Substitution from (3.18) shows that this reduces to

$$(\partial A_0/\partial s+A_0/2s)I=0, \tag{3.23}$$

and is satisfied in view of (3.9). We may set

$$(H_1, F_1, K_1)=\frac{\partial A_0}{\partial s}(h_1, f_1, k_1)+\frac{A_0}{s}(h_2, f_2, k_2), \tag{3.24}$$

where the functions $h_j(z)$, $f_j(z)$, $k_j(z)$, $j=1, 2$, satisfy equations of the form (3.7)–(3.8) but with the three zero right-hand sides replaced by terms involving the basic eigenfunctions and their derivatives. It can be shown that

$$h_1=-\partial h/\partial\alpha \quad\text{and}\quad h_2=-\tfrac{1}{2}\partial h/\partial\alpha. \tag{3.25}$$

Let $H=\partial h/\partial\alpha$; from (3.1)–(3.11)

$$L[H]=-2\alpha_c\left\{\left(\mathrm{D}^2-\alpha_c^2\right)^3+(T+R)\mathrm{D}^2-2\alpha_c^2 R\right\}h, \tag{3.26}$$

$$\left.\begin{aligned}&\mathrm{D}^2H=H=0,\\ &\mathrm{D}\left(\mathrm{D}^2-\alpha_c^2\right)H=2\alpha_c h,\\ &\mathrm{D}^3\left(\mathrm{D}^4-3\alpha_c^2\mathrm{D}^2+R_c+2\alpha_c^4\right)H=2\alpha_c\mathrm{D}^3\left(3\mathrm{D}^2-4\alpha_c^2\right)h,\end{aligned}\right\}\quad \text{at } z=0, 1 \tag{3.27}$$

and this system is again solved by use of the Runge–Kutta scheme. The particular solutions for w_2, v_2, θ_2 corresponding to the terms $E_{1,2,3,4}$ in (3.16) were also calculated numerically.

At order ϵ^3 ($i=3$), the integral constraints equivalent to (3.19), (3.20), (3.21) are only satisfied if

$$a_1\frac{\partial A_0}{\partial \bar{\tau}} - a_2 A_0 |A_0|^2 - a_3 s^2 A_0 - \frac{a_4}{s}\frac{\partial A_0}{\partial s} - a_5\frac{\partial^2 A_0}{\partial s^2} + \beta\, a_6 A_0 = 0, \tag{3.28}$$

where the coefficients a_i ($i=1,\ldots,6$) are found from appropriate numerical integrations (Fig. 2). It can be shown [7] that the coefficients a_3, a_4, an a_5 are related by

$$a_4 = a_5, \qquad a_4 = -4a_3, \tag{3.29}$$

providing a partial check on the numerical calculation. It should be noted that the coefficient of the nonlinear term a_2, changes sign for different values of the Prandtl number and Taylor number (Table 1, Fig. 3) so that, as in the stress-free case [10,5] subcritical instability can occur.

In view of (3.29), the transformation

$$A = A_0 |\zeta|^{1/2} s^{1/2}\, \mathrm{e}^{\mathrm{i}\lambda_0}, \tag{3.30}$$

can be used to reduce the amplitude equation (3.28) to the form

$$\frac{\partial A}{\partial \tau} - \frac{\partial^2 A}{\partial s^2} - A + \frac{1}{s} A|A|^2 \operatorname{sgn}(\xi) = 0 \tag{3.31}$$

where

$$\xi = -a_2/a_4, \qquad \tau = (a_4/a_1)\bar{\tau}, \qquad \beta = -a_4/a_6 \tag{3.32}$$

and λ_0 is an arbitrary constant. In order to consider the subcritical case we take $\beta = a_4/a_6 < 0$ and this changes the sign of the term A in (3.31).

4. Asymptotic analysis for $T \gg 1$

The numerical results for the critical Rayleigh number and wave number and the coefficients of the amplitude equation (Fig. 2) can be checked to some extent by comparison with an asymptotic approach for high Taylor numbers (Fig. 4.)

It emerges that there are three distinct horizontal layers when $T \gg 1$. Further, symmetry properties about $z=\frac{1}{2}$ can be used to restrict attention to $0 \leqslant z \leqslant \frac{1}{2}$. The inner boundary layer (region III) has the familiar scaling associated with small Ekman number flow near a rigid boundary, the thickness, in terms of T, being $\mathrm{O}(T^{-1/4})$. It also emerges that there is a middle layer (region II) where $z = \mathrm{O}(T^{-1/6})$ and finally, concentration of viscous action into these narrow layers means that elsewhere the fluid behaves in an essentially inviscid manner (region I). We shall solve equations (3.4)–(3.5) in these three different regions and use the method of matched asymptotic expansions to connect the solutions in the various regions.

We set

$$\left.\begin{aligned} \alpha &= \alpha_0 T^{1/6} + \alpha_1 T^{1/12} + \cdots, \\ R &= R_0 T^{2/3} + R_1 T^{7/12} + \cdots, \end{aligned}\right\} \tag{4.1}$$

where α and R are the wavenumber and Rayleigh number respectively and α_i, R_i ($i=0,1$) are unknown values which have to be found. The leading order scalings in (4.1) are suggested by comparable results for the stress-free case [6]. The second-order terms do not appear in the

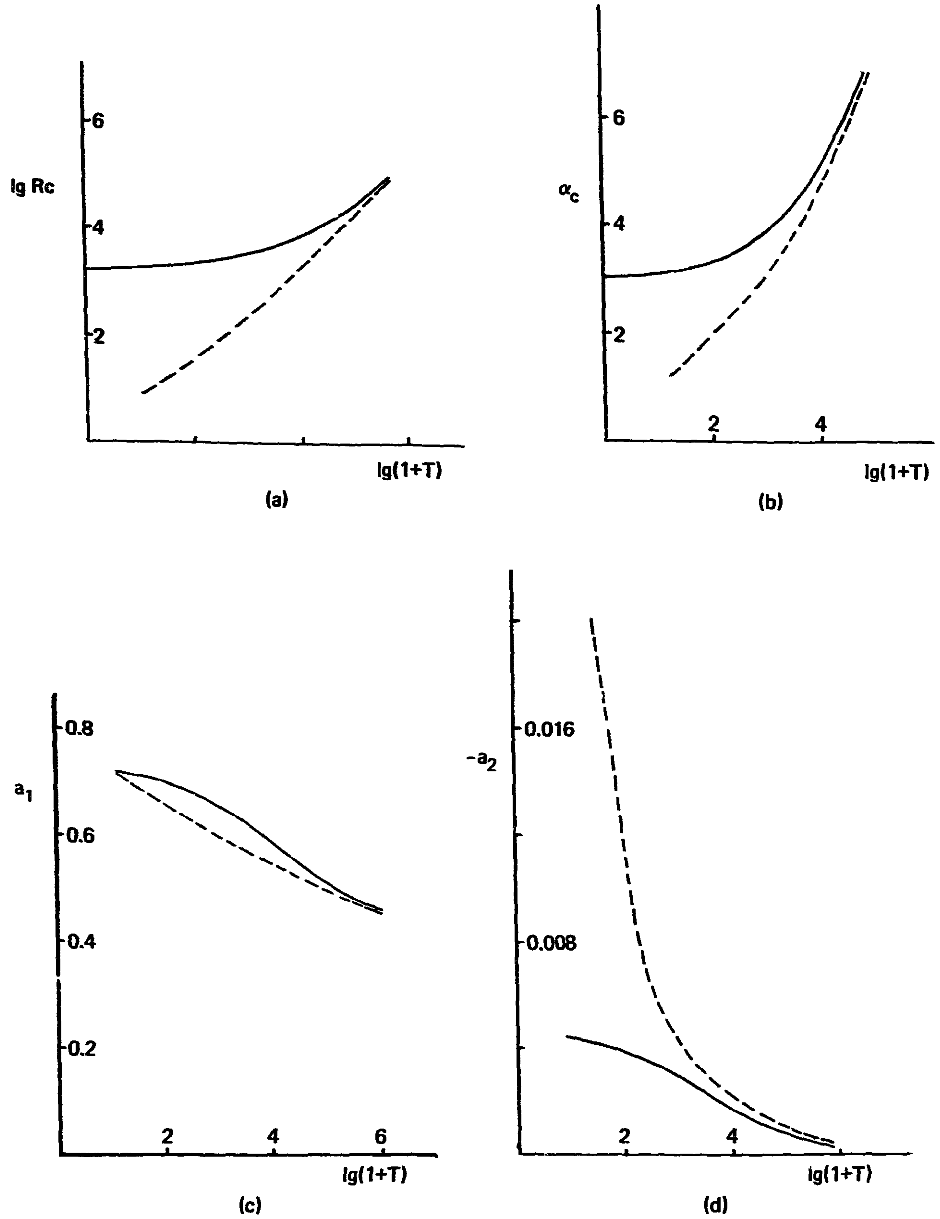

Fig. 2. Full lines show numerical results and dashed lines asymptotic theory for $T \gg 1$. (a) (R_c, T) relation, (b) (α_c, T) relation, (c) (a_1, T) elation for $\sigma = 1$, (d) (a_2, T) relation for $\sigma = 1$, (e) (a_3, T) relation for $\sigma = 1$, (f) (a_6, T) relation for $\sigma = 1$,

stress-free case, but in general, expansions in powers of $T^{-1/12}$ are suggested by the relative scaling of regions II and III.

The asymptotic expansions in the three different regions for $h(z)$, the solution of (3.10)–(3.11),

Table 1
Numerical results of a_2 (coefficient of nonlinear term in amplitude equation) for different Taylor numbers and Prandtl numbers, where $E-n=10^{-n}$, $n=1, 2, 3, \ldots, 6$.

σ	$T=10$	$T=100$	$T=500$	$T=1000$	$T=2000$	$T=5000$	$T=10000$	$T=30000$
0.1	−0.209 E−3	−0.145 E−4	0.393 E−4	0.142 E−3	0.206 E−3	0.187 E−3	0.130 E−3	0.561 E−4
0.2	−0.818 E−3	−0.65 E−4	−0.166 E−4	0.121 E−4	0.319 E−4	0.328 E−4	0.226 E−4	0.887 E−4
0.3	−0.587 E−3	−0.507 E−4	−0.27 E−4	−0.121 E−4	−0.52 E−5	0.42 E−5	0.272 E−5	0.106 E−5
0.4	−0.508 E−4	−0.458 E−4	−0.307 E−4	−0.207 E−4	−0.119 E−4	−0.581 E−5	−0.425 E−5	−0.296 E−5
0.5	−0.47 E−4	−0.43 E−4	−0.32 E−4	−0.247 E−4	−0.17 E−4	−0.104 E−4	−0.747 E−5	−0.438 E−5
0.6	−0.451 E−4	−0.42 E−4	−0.33 E−4	−0.269 E−4	−0.20 E−4	−0.129 E−4	−0.923 E−5	−0.515 E−5
0.7	−0.44 E−4	−0.418 E−4	−0.34 E−4	−0.28 E−4	−0.218 E−4	−0.145 E−4	−0.102 E−4	−0.56 E−5
0.8	−0.434 E−4	−0.41 E−4	−0.345 E−4	−0.29 E−4	−0.22 E−4	−0.154 E−4	−0.109 E−4	−0.69 E−5
0.9	−0.429 E−4	−0.41 E−4	−0.346 E−4	−0.297 E−4	−0.23 E−4	−0.16 E−4	−0.114 E−4	−0.61 E−5
1	−0.427 E−4	−0.409 E−4	−0.35 E−4	−0.302 E−4	−0.24 E−4	−0.167 E−4	−0.118 E−4	−0.629 E−5

are given by

$$\text{in (I)}: \quad h(z)=h_0+T^{-1/12}h_1+T^{-1/6}h_2+\cdots, \tag{4.2}$$

$$\text{in (II)}: h(z_1)=T^{-1/12}H_0+T^{-1/6}H_1+T^{-1/4}H_2+\cdots, \tag{4.3}$$

$$\text{in (III)}: h(z_2)=T^{-1/6}\overline{H}_0+T^{-1/4}\overline{H}_1+T^{-1/3}\overline{H}_2+\cdots, \tag{4.4}$$

where $z_1=T^{1/6}z$, $z_2=T^{1/4}z$.

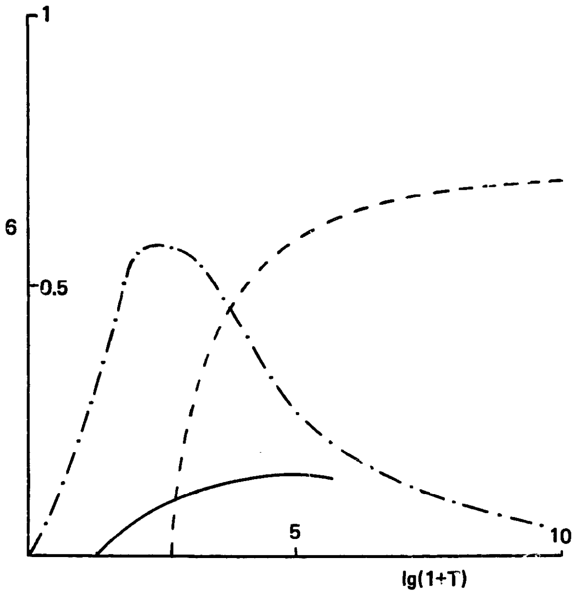

Fig. 3. Stability boundaries in the (σ, T) plane. Subcritical instability in the exchange case occurs below the boundary shown: ——— rigid-rigid case (present work); ·—·—· free-free case (Daniels [5] (1978). The effect of rigid horizontal planes is to substantially reduce the region of subcritical instability. In the free-free case overstability occurs below the boundary — — — (Chandrasekhar [2] 1961); the complete corresponding boundary in the rigid-rigid case is not available although it is known for example that there is a slight preference for overstability when $\sigma=0.025$, $T=10^4$ (Chandrasekhar [2]).

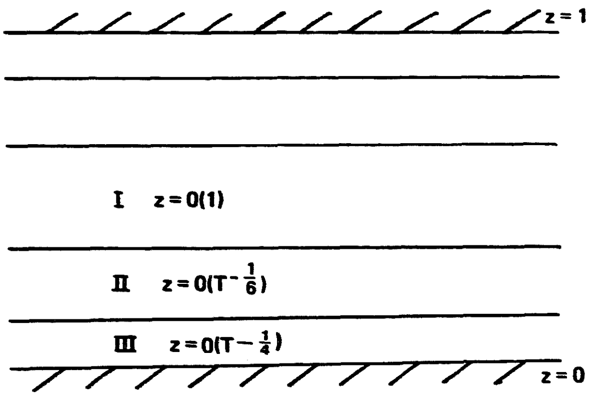

Fig. 4. Structure of flow between two rigid planes.

From (3.10)–(3.11) and (4.1)–(4.4), for the leading order terms we find that

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}z^2} - \alpha_0^2\left(\alpha_0^4 - R_0\right)\right\} h_0 = 0,$$
$$\frac{\mathrm{d}^2}{\mathrm{d}z_1^2}\left(\frac{\mathrm{d}^2}{\mathrm{d}z_1^2} - \alpha_0^2\right) H_0 = 0, \qquad \frac{\mathrm{d}^4}{\mathrm{d}z_2^4}\left(\frac{\mathrm{d}^4}{\mathrm{d}z_2^4} + 1\right) \bar{H}_0 = 0, \tag{4.5}$$

$$\bar{H}_0 = \bar{H}_0^{(n)} = 0, \quad n = 2, 3, 7 \quad \text{at } z_2 = 0, \tag{4.6}$$

where $\bar{H}_0^{(n)} = \mathrm{d}^n \bar{H}_i / \mathrm{d}z_2^n$. In view of the normalization condition (3.12) and the boundary conditions imposed by matching between II, I and III, the leading order structure is now given by

$$h_0 = \sin \pi z, \quad H_0 = c\left(1 - \mathrm{e}^{-\alpha_0 z_1}\right) \quad \text{and} \quad \bar{H}_0 = \alpha_0 c z_2 \tag{4.7}$$

where c remains to be found, and

$$\pi^2 + \alpha_0^2\left(\alpha_0^4 - R_0\right) = 0. \tag{4.8}$$

In region I at second order, we find that

$$h_1 = -\left(\alpha_0 / 2\pi\right)\left\{\alpha_1\left(6\alpha_0^4 - 2R_0\right) - R_1 \alpha_0\right\}\left(z - \tfrac{1}{2}\right) \cos \pi z, \tag{4.9}$$

and matching between region I and region II, implies that

$$c = \left(\alpha_0 / 4\pi\right)\left\{\alpha_1\left(6\alpha_0^4 - 2R_0\right) - R_1 \alpha_0\right\}. \tag{4.10}$$

Consideration of the solution to order $T^{-5/12}$ in region III and application of the boundary conditions at the wall. ($z = 0$) lead to the determination of c as

$$c = \pi / \sqrt{2}\,\alpha_0^2. \tag{4.11}$$

Details of this calculation are given by Golbabai [7]. If we now restrict attention to the situation in which α and R take their critical values, we also have $\mathrm{d}R/\mathrm{d}\alpha = 0$, and this provides a second relation between α_j, R_j ($j = 0, 1$). Use of (4.8), (4.10) and (4.11) then fixes the values of α_j, R_j ($j = 0, 1$) uniquely as

$$\alpha_0 = \left(\tfrac{1}{2}\pi^2\right)^{1/6}, \qquad R_0 = 3\alpha_0^4,$$
$$\alpha_1 = -\pi^2 \sqrt{\tfrac{2}{3}}\, \alpha_0^7, \qquad R_1 = -4^2 \pi / \alpha_0^4 \sqrt{2}. \tag{4.12}$$

Thus the first two terms in the asymptotic expansions of the critical Rayleigh number and wave number are found.

It is noted that similar expansions are derived by Homsy and Hudson [8], although their approach avoids consideration of the middle layer which is essentially a thermal boundary layer in which the vertical velocity f, is unchanged at leading order. Of course the asymptotic expansions of the other related functions can also be found from the known form of $h(z)$. The leading contributions to the amplitude equation coefficients come from the solution in region I, where

$$h = \sin \pi z - \left[(2\pi)^{1/3}\sqrt{2}\left(z - \tfrac{1}{2}\right) \cos \pi z\right] T^{-1/12} + \cdots . \tag{4.13}$$

The asymptotic expansions of these coefficients are compared with the numerical results in Fig. 2.

5. Inner solution and matching at the axis of rotation

For stress-free horizontal boundaries the amplitude equation is also of the form (3.31) and is, in fact, of the same form as that derived by Brown and Stewartson [1] for the nonrotating problem with stress-free boundaries. A linear inner solution for the stress-free rotating case of the same form as that found by Brown and Stewartson can be constructed in terms of the zeroth-order Bessel function J_0:

$$\theta = \left(aJ_0(\alpha_c r) + brJ_0'(\alpha_c r)\right) \sin \pi z, \tag{5.1}$$

although the characteristic equation relating α_c and R_c now depends on the Taylor number T, with $R_c = 3(\alpha_c^2 + \pi^2)^2$, and

$$\alpha_c^2 = \left(\tfrac{1}{2}\pi\right)^{-2/3}\left\{\left(\tfrac{1}{2}\pi^4 + T + T^{1/2}\left[\pi^4 + T\right]^{1/2}\right)^{1/3} + \left(\tfrac{1}{2}\pi^4 + T - T^{1/2}\left[\pi^4 + T\right]^{1/2}\right)^{1/3}\right\} - \tfrac{1}{2}\pi^2. \tag{5.2}$$

Thus matching between the inner and outer zones in the stress-free rotating problem follows exactly as in the nonrotating problem studied by brown and Stewartson and leads to a boundary condition for A at $s = 0$ of the form $A(0, \tau) = 0$. We establish that a similar condition applies in the case of rigid boundaries. Following the method used by Brown and stewartson [1], in the neighbourhood of $r = 0$ we work in terms of the variables r and τ and consider a solution of (2.1)–(2.5) in which the nonlinear terms are neglected. One solution is given by

$$\theta = J_0(\alpha_c r) h \tag{5.3}$$

where $h(z)$ is the solution of (3.10)–(3.11), and a second solution is

$$\theta = HJ_0(\alpha_c r) + rhJ_0'(\alpha_c r), \tag{5.4}$$

where $H(z) = \partial h/\partial\alpha$ is the solution of (3.26)–(3.27). Discarding other solutions which are exponentially large as $r \to \infty$, the general solution for θ in the inner zone may now be written in the form

$$\theta_I = \lambda J_0(\alpha_c r)h + \mu\left\{HJ_0(\alpha_c r) + rJ_0'(\alpha_C r)h\right\}. \tag{5.5}$$

In (5.5), θ_I denotes the inner solution and we use θ_O to denote the outer solution described in

Section 3. In order to match these solutions we need the behaviour of the amplitude function A (which satisfies 3.31), as $s \to 0$, which is found to be

$$A \sim a + bs + \operatorname{sgn}(\xi) a |a|^2 s \ln s + O(s^2 \ln s), \quad (s \to 0) \tag{5.6}$$

where a and b are functions of τ. In (3.30) it is convenient to take $\lambda_0 = \frac{1}{4}\pi$ and then from (3.3) and (3.6) the other solution is given by

$$\theta_0 = \left(\frac{\epsilon}{r|\xi|}\right)^{1/2} \left\{(\cos \hat{r} + \mathrm{i} \sin \hat{r}) Ah + \epsilon(\cos \hat{r} + \mathrm{i} \sin \hat{r}) \frac{\partial A}{\partial s} H + \cdots \right\} + \text{c.c}, \tag{5.7}$$

where c.c denotes complex conjugate and $s = \epsilon r$, $\hat{r} = \alpha_c r - \frac{1}{4}\pi$. Now the asymptotic expansion of (5.5) for large r is

$$\theta_1 \sim \left(\frac{2}{\pi \alpha_c r}\right)^{1/2} \left\{\left(\lambda - \frac{3\mu}{8\alpha_c}\right) h \cos \hat{r} + \mu(H \cos \hat{r} - rh \sin \hat{r})\right\}. \tag{5.8}$$

Thus from (5.6) and comparing (5.7) and (5.8) se see that a match of the terms $h/r^{1/2})$ $3c\,\hat{r}$, $hr^{1/2} \cos \hat{r}$, $(h/r^{1/2}) \sin \hat{r}$, $hr^{1/2} \sin \hat{r}$ is secured if, respectively,

$$\left.\begin{aligned}
&a - a^* = 0,\\
&(a + a^*)\epsilon^{1/2} - (2|\xi|/\pi\alpha_c)^{1/2}(\lambda - 3\mu/8\alpha_c) = 0,\\
&b + b^* + |a|^2(a + a^*) \operatorname{sgn}(\xi) \ln|\xi| = 0,\\
&\mathrm{i}(b - b^*)\epsilon^{3/2} - \mu(2|\xi|/\pi\alpha_c)^{1/2} = 0.
\end{aligned}\right\} \tag{5.9}$$

It should be noted that the terms in $(H/r^{1/2}) \sin \hat{r}$, $(H/r^{1/2}) \cos \hat{r}$ match automatically and from the matching conditions (5.9), we find that a is real and

$$a^3 = -\tfrac{1}{2}\operatorname{sgn}(\xi)(b + b^*)/\ln \epsilon. \tag{5.10}$$

If we suppose that $|b| = O(1)$, consistent with the assumption that $|A|$ is O(1) when $s = O(1)$, it follows that

$$a = O\big((-\ln \epsilon)^{-1/3}\big). \tag{5.11}$$

Thus equation (3.31) should be solved in the form of a series for A in ascending powers of $(-\ln \epsilon)^{-1/3}$ with the first approximation having

$$\alpha = 0. \tag{5.12}$$

Thus in the limit $\epsilon \to 0$, the boundary condition for A is given by

$$A(0, \tau) = 0. \tag{5.13}$$

This condition is the same as that derived by Brown and Stewartson [1] for the case of a nonrotating layer with stress-free boundaries. The solution of (3.31) subject to (5.13) is discussed in Section 6.

6. Solution of the amplitude equation

In this section we consider numerical and asymptotic solutions of the amplitude equation (3.31). Where for supercritical Rayleigh numbers $A(s)$ satisfies

$$\begin{aligned}
&\mathrm{d}^2A/\mathrm{d}s^2 + A - \operatorname{sgn}(\xi)A/s = 0,\\
&A(0) = 0.
\end{aligned} \tag{6.1}$$

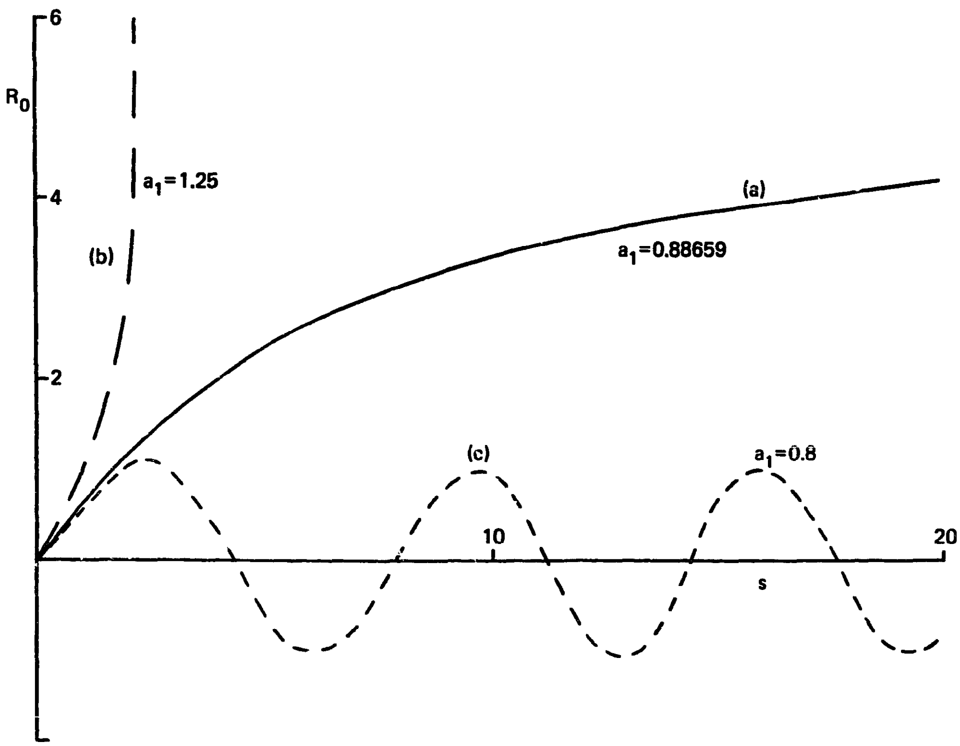

Fig. 5. Profiles of A for the supercritical solution with $\xi > 0$.

Let $\xi > 0$, then A can be expanded as a power series in s as

$$A = a_1\left(s - \tfrac{1}{6}s^3 + \cdots\right) \quad (s \to 0) \tag{6.2}$$

where a_1 is an arbitrary constant and

$$\mathrm{d}A/\mathrm{d}s = a_1 \quad (s \to 0). \tag{6.3}$$

If we regard a_1 as known, (6.1)–(6.3) constitutes an initial value problem for $A(s)$. A fourth-order Runge–Kutta scheme was used and several profiles of A for different values of a_1 are given in Fig. 5. If a_1 is too large, A tends to infinity at a finite value of s, while if a_1 is too small the solution for A passes through zero and oscillates as $s \to \infty$. The required solution, which is likely to represent a stable flow pattern is the intermediate one for which

$$A \sim s^{1/2} \quad \text{as } s \to \infty. \tag{6.4}$$

This corresponds to a uniform amplitude motion at large distance from the axis of rotation. The required value of a_1 was found to be

$$a_1 = 0.8866. \tag{6.5}$$

We now discuss in some detail the asymptotic expansion of A when s is large. As can be seen from Fig. 5, there are three possible behaviours of A which correspond to the curves (a), (b), (c) in Fig. 5. The appropriate asymptotic forms of each curve are as follows;

(a) $$A \sim s^{1/2} - \tfrac{1}{8}s^{-3/2} \quad (s \to \infty), \tag{6.6}$$

(b) $$A \sim \sqrt{2}\, s_0^{1/2}(s_0 - s)^{-1} \quad (s \to s_0), \tag{6.7}$$

(c) $$A \sim a \sin(f(s)) \quad (s \to \infty) \tag{6.8}$$

in (6.8), a is an arbitrary constant and $f(s) \to s$ as $s \to \infty$, indicating a balance between the linear terms in (6.1). However, the nonlinear term does have a significant effect on the asymptotic structure and it emerges that $f(s)$ and $A(s)$ may be written as

$$\left.\begin{aligned} f(s) &= s + b \ln s + c, \\ A(s) &= a \sin(f) + (1/s) h_1(f) + \cdots, \end{aligned}\right\} \quad (s \to \infty). \tag{6.9}$$

From (6.1) and (6.9), it is found that

$$b = -\tfrac{3}{8} a^2, \tag{6.10}$$

and h_1 can be determined to give

$$A(s) \sim a \sin f + (a^3/s)\{\tfrac{3}{16} \sin f + \tfrac{51}{256} a^2 \cos f + \tfrac{1}{32} \sin 3f\} \tag{6.11}$$

where a is an arbitrary constant. If in (6.1), $\xi < 0$, then all solutions for different values of a_1 oscillate as $s \to \infty$, and the result (6.9)–(6.11) are modified to become

$$A(s) \sim a \sin f + (a^3/s)\{-\tfrac{3}{16} \sin f + \tfrac{51}{256} a^2 \cos f - \tfrac{1}{32} \sin 3f\} \tag{6.12}$$

as $s \to \infty$, where

$$f(s) = s + \tfrac{3}{8} a^2 \ln s + c. \tag{6.13}$$

For subcritical Rayleigh numbers, $A(s)$ satisfies

$$\left.\begin{aligned} &\mathrm{d}^2A/\mathrm{d}s^2 + A - \mathrm{sgn}(\xi) A^3/s = 0, \\ &A(0) = 0. \end{aligned}\right\} \tag{6.14}$$

Numerical solutions can be obtained using the previous method, but now all solutions A, for different values of a_1, are unbounded when $\xi < 0$; the solution for A becomes unbounded in the form (6.7) at a finite value of s. For the case $\xi < 0$, the numerical solutions are plotted by Golbabai [1] but it seems likely that all such solutions correspond to unstable motions. In general, the asymptotic form of A as $s \to \infty$ is given by

$$A \sim s^{1/2} + a \cos(\sqrt{2}\, s + b), \tag{6.15}$$

where a, b are arbitrary constants, although there is a family of eigenfunctions which decay exponentially as $s \to \infty$,

$$A \sim c\left(\mathrm{e}^{-s} - \tfrac{1}{8} c\, \mathrm{e}^{-3s}\right),$$

where c is a constant.

7. Discussion

The structure of an axisymmetric finite amplitude convection in a rotating layer with rigid boundaries, when the system is subject to the change of stabilities, is described; and it is shown that, for certain ranges of the speed of rotation and Prandtl number of the fluid, subcritical instabilities in possible as in the model problem with stress-free boundaries considered by Veronis [10].

The result of the analysis in Section 5 is the boundary condition (6.1) which is the same as that in the case of a nonrotating layer with stress-free boundaries, so that the main feature of [1] applied to the rotating case also. Hence the amplitude function A can be expanded in the form of a series in ascending powers of $(-\ln \epsilon)^{-1/3}$.

Here we restricted ourselves to the assumption that (5.1), implies $a = 0$ in the limit as $\epsilon \to 0$. It is hoped that future work will consider $a = O(-\ln \epsilon)^{-1/3}$.

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